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Condensation phenomena of a conserved-mass aggregation model on weighted complex networks

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We investigate the condensation phase transitions of the conserved-mass aggregation (CA) model on weighted scale-free networks (WSFNs). In WSFNs, the weight w_{ij} is assigned to the link between the nodes i and j. We consider the symmetric weight given by $w_{ij}=(k_i k_j)^{\alpha}$. On WSFNs, we numerically show that a certain critical α_c exists below which the CA model undergoes the same type of condensation transitions as those of the CA model on regular lattices. However, for $\alpha \geq \alpha_c$, the condensation always occurs for any density ρ and ω. We analytically find $\alpha_c=(\gamma-3)/2$ on the WSFN with the degree exponent γ. To obtain α_c, we analytically derive the scaling behavior of the stationary probability distribution P_k^{∞} of finding a walker at nodes with degree k, and the probability $D(k)$ of finding two walkers simultaneously at the same node with degree k. We find $P_k^{\infty} \sim k^{\alpha+1-\gamma}$ and $D(k) \sim k^{2(\alpha+1)-\gamma}$, respectively. With P_k^{∞}, we also show analytically and numerically that the average mass $m(k)$ on a node with degree k scales as $k^{\alpha+1}$ without any jumps at the maximal degree of the network for any ρ as in SFNs with $\alpha=0$.

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I. INTRODUCTION

A wide variety of mass transport systems ranging from traffic flow to polymer gels [1–8] exhibit nonequilibrium condensation phenomena. These include basic microscopic dynamics ubiquitous in nature such as aggregation, fragmentation, and diffusion. The nonequilibrium steady states of these systems are classified into two types of phase, the so-called fluid and condensed phases. A finite fraction of total particles condenses on a single site in the condensed phase. In the fluid phase, the particle number on each site fluctuates around the density of total particles (ρ) without condensation. As the rates of these processes vary, a condensation phase transition between the two phases may take place at a certain critical density ρ_c.

One of the simplest mass transport models exhibiting condensation transitions is the conserved-mass aggregation (CA) model [9–13]. The CA model evolves via diffusion, chipping, and aggregation upon contact, processes that arise in a variety of phenomena such as polymer gels [4], the formation of colloidal suspensions [5], river networks [6,7], and clouds [8]. In one-dimensional CA models, the mass m_i of site i moves either to site $i-1$ or to site $i+1$ with unit rate, and then $m_i \to 0$ and $m_{i\pm1} \to m_{i\pm1}+m_i$. With rate ω, unit mass chips off from site i and moves to one of the nearest-neighboring sites; $m_i \to m_i-1$ and $m_{i\pm1} \to m_{i\pm1}+1$. The generalization to higher dimensions is straightforward. As total masses are conserved, the conserved density ρ and the rate ω determine the phase of the CA model. The $\omega=\infty$ case corresponds to the well-known zero-range process (ZRP) with a constant hopping rate [14–16].

The existence of the condensation transitions in CA model depends on the symmetry of movement, the constraints of the diffusion rate, and the underlying network structure [10–13]. In the symmetric CA (SCA) model [9,10] in which the diffusion and chipping direction are unbiased, condensation transitions take place at a certain ρ_c. The steady state properties of the SCA model are exactly described by the mean field theory [10]. The single-site mass distribution $P(m)$ was shown to undergo phase transitions on regular lattices [9]. For a fixed ω, as ρ is varied across the critical density $\rho_c(\omega)$, the behavior of $P(m)$ for large m was found to be [9]

$$P(m) \sim \begin{cases} e^{-m/m^*}, & \rho < \rho_c(\omega), \\ m^{-\tau}, & \rho = \rho_c(\omega), \\ m^{-\tau} + \text{infinite aggregate}, & \rho > \rho_c(\omega). \end{cases} \tag{1}$$

Mean field theory predicts $\rho_c(\omega)=\sqrt{\omega+1}-1$ and $\tau=5/2$ [9,10].

Recently, the CA model on unweighted scale-free networks (SFNs) with degree distribution $P(k) \sim k^{-\gamma}$ was studied to investigate the effect of the underlying network structure on the condensation transitions [13]. We call networks with equal weight on all links unweighted networks. On unweighted SFNs, the same type of condensation transitions as those of the SCA in regular lattices take place for $\gamma > 3$. However, for $\gamma \leq 3$, condensation always occurs for any density ρ (>0). It was shown that the existence of the transitions is directly related to the diffusive capture process on unweighted SFNs [13,26].

On the other hand, most real-world networks exhibit not only a heterogeneous distribution of degree, but also a heterogeneous distribution of weights [17–19]. Weights assigned to the links characterize the interaction strengths between nodes. There have been various attempts to understand the underlying mechanism and scale-free behaviors of empirically observed weighted networks [20]. Also there have been attempts to understand the effect of heterogeneous weights on various dynamics such as synchronization, dynamics of random walks, transport and percolation, condensation of zero-range processes, and equilibrium and nonequilibrium phase transitions [21–24]. These studies showed that dynamical properties are modified and exhibit nontrivial dependence on the strength of the weight. In this paper, as a generalization of our study of the CA model on complex networks, we investigate the effect of both heterogeneous degree and weight on the condensation phenomena of the CA model on weighted networks.

The weight w_{ji} represents the weight of a link from the node i to j. In general, the strength s_i of the node i scales with the degree k_i as $s_i \sim k_i^{\alpha}$. The exponent α varies with network structures [19,20]. Thus it is natural to take the weight w_{ji} as $w_{ji} \sim s_i s_j \sim (k_i k_j)^{\alpha}$.

In this paper, we study the condensation transitions of the CA on weighted SFNs (WSFNs) with degree distribution $P(k) \sim k^{-\gamma}$ and the symmetric weight $w_{ji} = (k_i k_j)^{\alpha}$. As in one dimension, the diffusion of the whole mass and the fragmentation of unit mass occur with unit rate and the rate ω, respectively. In addition, a mass moves from a node i to j with hopping rate proportional to $w_{ji}/\Sigma_j w_{ji}$. We found that a certain critical α_c exists below which the condensation transitions take place. However, for $\alpha \geq \alpha_c$, the condensation always occurs for any density $\rho > 0$. To find α_c as a function of the degree exponent γ, one needs the steady state distribution P_k^{∞} of finding a walker at nodes with degree k on the WSFN. P_k^{∞} gives the capture probability $D(k)$ with which two walkers meet at a node with degree k. We analytically derived P_k and $D(k)$, and finally obtained $\alpha_c = (\gamma - 3)/2$.

This paper is organized as follows. In Sec. II, we discuss the condensation transitions of the CA model on WSFNs. To verify the existence of α_c, we investigate the steady state properties of a single walker and the diffusive capture process on WSFNs in Secs. III and IV. We discuss the behavior of the average mass $m(k)$ of a node with degree k in Sec. V, and finally summarize our results in Sec. VI.

II. CA MODEL ON WSFNS WITH SYMMETRIC WEIGHTS

We consider the CA model on a WSFN with the weight w_{ij} from node j to i defined as $w_{ij} = (k_i k_j)^{\alpha}$. For the construction of the WSFN, we first construct an unweighted static SFN with N nodes and K links [27]. The degree k_i of a node i is defined as the number of its links connected to other nodes. The average degree of a node $\langle k \rangle$ is given as $\langle k \rangle = 2K/N$. The degree distribution $P(k)$ of SFNs is a power-law distribution $P(k) \sim k^{-\gamma}$. In the static model [27], it is desired to use large $\langle k \rangle$ to construct fully connected networks. In simulations, we use $\langle k \rangle = 4$. After that, we assign a weight $w_{ij} = (k_i k_j)^{\alpha}$ to the link between nodes i and j. Thus the hopping probability of a mass from node i to one of its linked neighbors j is $T_{ji} = k_i^{\alpha} k_j^{\alpha} / \Sigma_{\langle m \rangle} k_m^{\alpha} k_i^{\alpha} = k_j^{\alpha} / \Sigma_{\langle m \rangle} k_m^{\alpha}$. $\Sigma_{\langle m \rangle}$ denotes the sum over the linked neighbors of node i.

Each node has an integer number of particles, and the mass on a node is defined as the number of particles on the node. Initially M particles are randomly distributed on N nodes with given conserved density $\rho = M/N$. Next a node i is chosen at random and one of the following events occurs.

(i) Diffusion: With unit rate, the whole mass m_i of node i moves to one of the linked neighbors j with probability T_{ji}. Then aggregation takes place; $m_i \to 0$ and $m_j \to m_j + m_i$.

(ii) Chipping: With rate ω, a unit mass moves to a linked neighbor j with the probability T_{ji}, and then aggregation takes place, i.e., $m_i \to m_i - 1$ and $m_j \to m_j + 1$.

The $\omega = \infty$ case corresponds to a ZRP with constant chipping rate on a WSFN [16].

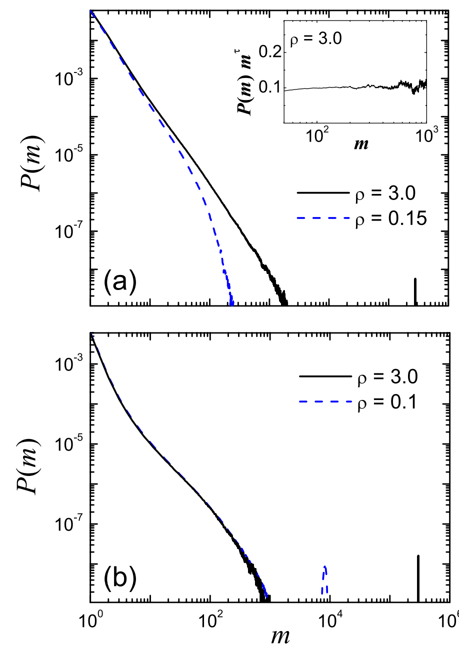

FIG. 1. (Color online) $P(m)$ for $\gamma = 4.0$ with $\alpha = 0.05$ (a) and 1.0 (b). The inset of (a) shows the scaling plot $m^{\tau} P(m)$ with $\tau = 2.38$ when $\rho = 3.0$.

We perform Monte Carlo simulations with random initial mass distribution on WSFNs with $\gamma = 2.7$ and 4.0. We set $\omega = 1$ and the network size $N = 10^5$ with $\langle k \rangle = 4$. We measure the single-node mass distribution $P(m)$ in the steady states.

In Fig. 1, we plot $P(m)$ for $\gamma = 4.0$ with two different α, $\alpha = 0.05$ and 1.0. $P(m)$ exhibits quite different behavior according to the value of α. For $\alpha = 0.05$ [Fig. 1(a)], $P(m)$ decays exponentially without aggregates for sufficiently low density $\rho = 0.15$. On the other hand, for sufficiently high density, $\rho = 3.0$, an aggregate forms with a power-law-decaying background mass distribution. This means that the condensation transition takes place at a certain critical density ρ_c (> 0). Hence $P(m)$ follows Eq. (1). Since in unweighted SFNs, i.e., for $\alpha = 0$, condensation phase transitions take place for $\gamma > 3$ [13], one may expect condensation transitions for very small α. Based on the following steps, we estimate ρ_c and the exponent τ.

In the condensed phase, the total density ρ is written as $\rho = \rho_c + \rho_{\infty}$, where ρ_{∞} is the density of an aggregate. Since ρ is given as $\rho = \int_1^{\infty} m P(m) dm$, one can estimate ρ_c from $\rho_c = \int_1^{m_o} m P(m) dm$, where the upper bound m_o is the cutoff mass at which the background distribution terminates. Using this method, we estimate $\rho_c = 0.218$. We estimate the exponent τ from the scaling plot $m^{\tau} P(m)$ using the $P(m)$ for $\rho = 3.0$ [inset of Fig. 1(a)]. Since the background distribution does not change for $\rho \geq \rho_c$, we use the $P(m)$ for $m \leq m_o$ for the scaling plot. We estimate $\tau = 2.38(5)$.

On the other hand, for $\alpha = 1.0$ [Fig. 1(b)], $P(m)$ shows completely different behavior. The condensation takes place with an exponentially decaying background distribution for both low and high density, $\rho = 0.1$ and 3.0. Therefore we conclude that the condensation always occurs for any nonzero density, so the system is always in the condensed phase without any transitions for $\alpha = 1.0$. The two different behav-

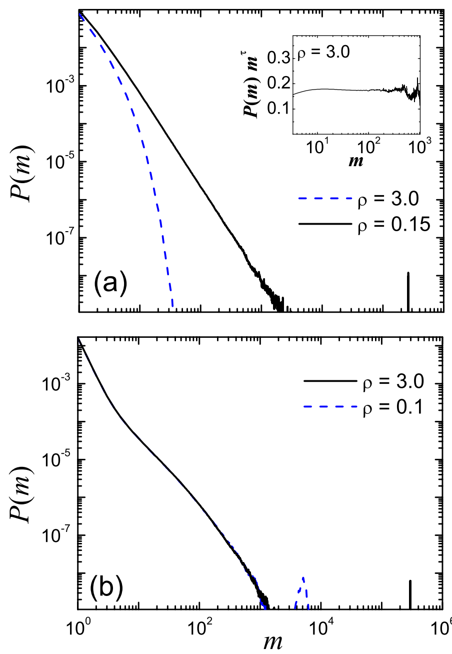

FIG. 2. (Color online) $P(m)$ for $\gamma=2.7$ with $\alpha=-1.0$ (a) and -0.05 (b). The inset of (a) shows the scaling plot $m^{\tau}P(m)$ with $\tau=2.46$ when $\rho=3.0$.

iors of $P(m)$ for $\alpha=0.05$ and 1.0 indicate that a crossover α (α_c) should exist in the range $0.05<\alpha<1.0$ for $\gamma=4.0$. A system undergoes the condensation transition for $\alpha<\alpha_c$, while ondensation always occurs without a transition for $\alpha \geq \alpha_c$.

Similarly, for $\gamma=2.7$, $P(m)$ also exhibits different behavior according to the value of α. The difference from the $\gamma=4.0$ case is that the condensation transitions are observed for a negative α. We observe the condensation transitions for $\alpha=-1.0$ [Fig. 2(a)]. With the same method used in the $\gamma=4.0$ case, we estimate $\rho_c=0.4$ and $\tau=2.46(5)$, respectively. However, for $\alpha=-0.05$, the condensation is observed even for very low density $\rho=0.1$, which means that the system is always in the condensed phase for $\alpha=-0.05$ [Fig. 2(b)]. Therefore, the crossover α_c also exists for $\gamma=2.7$, but its value is negative, unlike in the $\gamma=4.0$ case. Together with the results for $\gamma=4.0$, we conclude that the crossover α_c exists for any γ (>2) and α_c varies with γ. In what follows, we discuss the existence of α_c and next the nature of the condensation for $\alpha<\alpha_c$.

First, the condensation phenomena of the CA model on WSFNs are similar to those on unweighted SFNs. On unweighted SFNs, condensation transitions exist for $\gamma>3$, while condensation always occurs for $\gamma\leq 3$ [13]. Hence the crossover γ is $\gamma_c=3$. Intriguingly, it was shown that the existence of transitions is determined by the survival probability of a diffusing prey chased by a diffusing predator, the so-called lamb-lion problem [28]. The reason is as follows.

In the limit $\rho\to 0$, let us assume that only an infinite aggregate exists. With the rate ω, unit mass is chipped off and moves around with unit rate. If the chipped mass meets again with the infinite aggregate within a finite time interval, then the infinite aggregate is stable against the chipping process. On the other hand, if the chipped mass and the infinite aggregate do not meet again within a finite time interval, then the infinite aggregate will disappear due to the repeated chipping processes. Therefore, the stability of the infinite aggregate is physically related to the capture process in which a diffusing lion (infinite aggregate) chases a diffusing lamb (chipped mass). For the unweighted SFNs with $\gamma\leq 3$, it was shown that the survival probability $S(t)$ of a lamb decays exponentially with finite lifetime $\langle T\rangle_\infty$ [13,26]. However, for $\gamma>3$, $S(t)$ is finite in the thermodynamic limit. The behavior of $S(t)$ implies that the condensation transition exist for $\gamma>3$ due to the stable fluid phase in the limit $\rho\to 0$, but only condensation exists for $\gamma\leq 3$. As a result, the asymptotic behavior of the survival probability of a lamb in the lamb-lion capture process determines the existence of condensation transitions on unweighted SFNs.

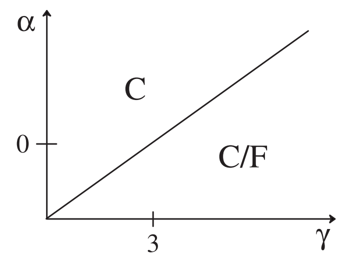

FIG. 3. α-γ phase diagram. The straight line denotes the critical line $\alpha_c=(\gamma-3)/2$. "C" denotes the condensed phase. "C/F" denotes the phase where the condensation transition occurs at a certain ρ_c.

Similarly, on WSFNs, the existence of the condensation transitions is also expected to depend on the survival probability $S(t)$ of a lamb. To see this, let us consider two limits $\alpha\to+\infty$ and $-\infty$ for a given γ. In the limit $\alpha\to+\infty$, a walker always moves to a node with a larger degree. Once a walker reaches the hub node with the maximal degree, the walker is trapped between the hub node and its linked nodes. As a result, a lion always captures a lamb at the hub node within a finite time interval. Hence, $S(t)$ should decay exponentially with a finite lifetime. On the other hand, in the limit $\alpha\to-\infty$, a walker is forced to reach nodes with the minimal degree. Due to the inhomogeneous structure, the nodes with the minimal degree are connected by nodes with larger degree. Hence, a walker is trapped between the node with minimal degree and its linked nodes in this limit. This means that a lion cannot always capture a lamb at some other node with minimal degree, so that $S(t)$ is finite.

From the behavior of $S(t)$ in the two opposite limits, there should be a crossover α_c. $S(t)$ is finite for $\alpha<\alpha_c$ and decays to zero for $\alpha\geq\alpha_c$. For the condensation phenomena, one expects no condensation transitions ($\rho_c=0$) for $\alpha\geq\alpha_c$ due to the finite lifetime of a lamb. Instead, condensation always occurs. On the other hand, condensation transitions occur for $\alpha<\alpha_c$. We analytically find $\alpha_c=(\gamma-3)/2$ for a given γ in Sec. IV. From $\alpha_c=(\gamma-3)/2$, one reads $\alpha_c=-0.15$ for $\gamma=2.7$ and $\alpha_c=0.5$ for $\gamma=4$. Our simulation results for $\gamma=4$ and 2.7 confirm the existence of α_c and also the sign of α_c for each γ. Figure 3 shows the corresponding schematic α-γ phase diagram.

Next, we discuss the critical behavior of the CA model on WSFNs. The CA model on any dimensional regular lattice and unweighted SFNs with $\gamma>3$ is well described by mean field theory [10,13]. On WSFNs, interestingly, the transitions take place even for $\gamma < 3$, which means that the transition nature is not affected by the inhomogeneity of the network structure. Since α_c diverges for $\gamma \to \infty$, the critical behavior of the CA model on SFNs with $\alpha < \alpha_c$ should be the same as that on random networks where α, i.e., the weight, has no special meaning due to the uniform degree distribution. As a result, one expects mean field critical behavior of the SCA model on a regular lattice. Our numerical estimates of τ, $\tau = 2.38(5)$ for $\gamma = 2.7$ and $\tau = 2.46(5)$ for 4.0, well agree with the mean field value $\tau = 5/2$. Therefore, we conclude that the critical behavior of the CA model for $\alpha < \alpha_c$ on WSFNs belongs to the universality class of the SCA model on a regular lattice.

In summary, for a fixed γ, there is a crossover weight exponent α_c. The CA model undergoes the same type of condensation transitions as those of the SCA model on a regular lattice for $\alpha < \alpha_c$, while condensation always takes place for nonzero density for $\alpha \geq \alpha_c$. To find α_c as a function of the degree exponent γ, one needs the steady state distribution P_k^∞ of finding a walker at nodes with degree k on the WSFN. In the next section, we derive P_k^∞ on the WSFN. In Sec. IV, we study the lamb-lion capture process on the WSFNs and finally find α_c using P_k^∞.

III. WALKS ON WSFNS WITH SYMMETRIC WEIGHTS

We consider a single walker on a weighted network with weight w_{ij}. The connectivity of the network is represented by the adjacency matrix $\mathbf{A}$ whose element $A_{ij} = 1$ if there is a link from a node j to i. Otherwise, $A_{ij} = 0$. We set $A_{ii} = 0$ conventionally. The degree k_i of a node i is given as $k_i = \Sigma_j A_{ji}$. Since we consider weighted networks with weight w_{ij}, we define the weighted adjacency matrix $\tilde{\mathbf{A}}$ as $\tilde{A}_{ij} = w_{ij} A_{ij}$.

The motion of a walker on the weighted network defined by the matrix $\tilde{\mathbf{A}}$ is a stochastic process in discrete time. We derive the stationary probability distribution P_i^∞ of a walker being at node i following the method of Ref. [25]. To set up the equation, we define the transition probability as follows. A walker at node i at time t selects one of its k_i linked nodes with hopping probability T_{ji}. Then, at time $t+1$, the walker moves to the selected node. The hopping probability T_{ji} from node i to j is then given as $T_{ji} = \tilde{A}_{ji} / \tilde{K}_i$, where $\tilde{K}_i = \Sigma_j \tilde{A}_{ji}$ is the strength of node i. As an initial condition, assume that the walker starts at the node q at time $t=0$. Then the recurrence relation of the transition probability P_{iq} of finding the walker at node i at time t is

$$P_{iq}(t+1) = \sum_l T_{il} P_{lq}(t). \tag{2}$$

Then the transition probability $P_{iq}(t)$ is written by iteration as

$$P_{iq}(t) = \sum_{l_1, \ldots, l_{t-1}} T_{il_{t-1}} \cdots T_{l_2 l_1} T_{l_1 q}. \tag{3}$$

For a symmetric $\tilde{\mathbf{A}}$ with $\tilde{A}_{ij} = \tilde{A}_{ji}$, one finds $\tilde{K}_q P_{iq}(t) = \tilde{K}_i P_{qi}(t)$ by comparing P_{qi} and P_{iq}. In the stationary state, the probability P_i^∞ of finding a walker at node i should be independent of the initial starting nodes, which gives $\tilde{K}_i P_q^\infty = \tilde{K}_q P_i^\infty$. Summing up over q, one finds

$$P_i^\infty = \tilde{K}_i / \mathcal{N}, \tag{4}$$

where $\mathcal{N} = \Sigma_{q=1}^N \tilde{K}_q = \Sigma_{q=1}^N \Sigma_{m=1}^N \tilde{A}_{mq}$. In weighted networks with symmetric weights, P_i^∞ is proportional to the strength of node i, i.e., the sum of the weights of the nearest-neighboring nodes. The same result was found in a recent study on the dynamics of random walks on growing weighted networks [22].

In this paper, we consider the symmetric weight w_{ij},

$$w_{ij} = (k_i k_j)^\alpha. \tag{5}$$

For the weight (5), P_i^∞ is not given as a simple form. Hence it is better to handle the distribution P_k^∞ of finding a walker at nodes with degree k. Using Eq. (4), one can see that

$$P_k^\infty = \sum_{i=1}^N P_i^\infty \delta_{k_i k} = \frac{1}{\mathcal{N}} \sum_{i=1}^N \sum_{j=1}^N A_{ji} (k_i k_j)^\alpha \delta_{k_i, k}. \tag{6}$$

To express the sum in Eq. (6) in terms of degree k, we arrange the sum as follows. Only terms with $k_i = k$ contribute nontrivially to the sum Σ_i and thus $NP(k)$ nodes with the degree k in a network have nontrivial contributions to the sum. The node with degree k has k linked neighbors whose degrees range from 1 to the maximal degree of the network $k_{\max}$. Hence, the number of nontrivial terms in the sum $\Sigma_{i=1}^N \Sigma_{j=1}^N$ is $NP(k)k$, which can be arranged in order of increasing degree. Then the double sum of Eq. (6) is written as $NP(k) k^{\alpha+1} [g(1)1 + g(2)2^\alpha + \cdots + g(k_{\max}) k_{\max}^\alpha]$, where $g(k')$ is the degree distribution of the node involved in such $NP(k)k$ terms. For large N, we approximate $g(k')$ to $P(k')$. Then P_k^∞ is approximately given as

$$\begin{aligned} P_k^\infty &= \frac{N}{\mathcal{N}} P(k) k^{\alpha+1} \int_{k_0}^{k_{\max}} P(k') k'^{\alpha} dk' \\ &= P(k) k^{\alpha+1} / \int_{k_0}^{k_{\max}} P(k') k'^{\alpha+1} dk'. \end{aligned} \tag{7}$$

On SFNs with degree distribution $P(k) \sim k^{-\gamma}$, the integral in the second equality is finite for $\alpha < \gamma - 2$. Hence we finally obtain P_k^∞ on the WSFN as

$$P_k^\infty \sim k^\sigma, \quad \sigma = \alpha + 1 - \gamma. \tag{8}$$

The exponent σ varies with α and γ, and also changes its sign. For $\alpha = \gamma - 1$, i.e., $\sigma = 0$, P_k^∞ is independent of the degree k so a walker is not affected by the inhomogeneity of the underlying network structure. While a walker performs biased walks to nodes with the larger degree for $\sigma > 0$, the direction of the bias is reversed for $\sigma < 0$. Since the exponent α is a free parameter, one can controls the direction of the bias for a given γ.

To check the scaling relation (8), we perform Monte Carlo simulations on WSFNs with $N = 10^5$ and the average degree $\langle k \rangle = 4$. In the steady states, we measure P_k^∞ for various α up to 2.6 for $\gamma = 2.7$ and 3.0 for $\gamma = 3.3$. Figure 4 shows the plot

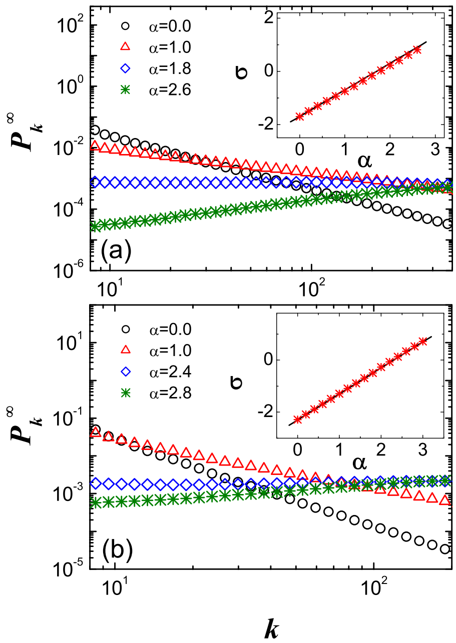

FIG. 4. (Color online) P_k^∞ and σ for $\gamma=2.7$ (a) and 3.3 (b). Insets show the relation (8) (solid line) and numerical estimates of σ (symbols).

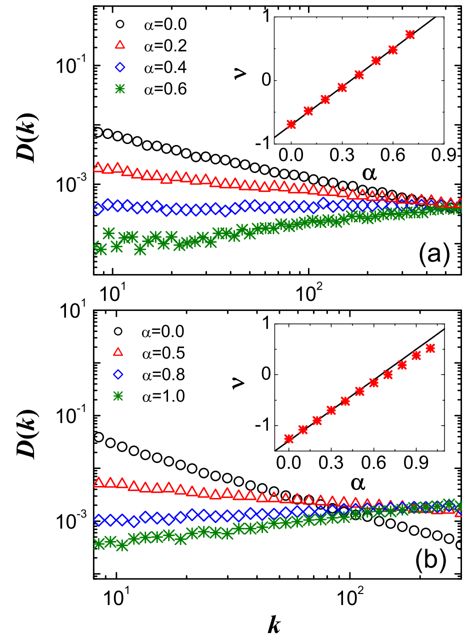

FIG. 5. (Color online) $D(k)$ and ν for $\gamma=2.7$ (a) and 3.3 (b). Insets show the relation of (10) (solid line) and numerical estimates of ν (symbols).

of P_k^∞ against k for several values of α. As shown, P_k^∞ scales as a power law with k. The inset in each panel shows the plot of σ against α. The simulation results agree well with the analytical prediction (8).

IV. CAPTURE PROCESS ON WSFNS WITH SYMMETRIC WEIGHTS

In this section, we consider the capture process or the lamb-lion problem on WSFNs with symmetric weights (5). A lamb and a lion are initially located separately on randomly selected two nodes. Then the probability $D(k)$ of finding two walkers at the same node with degree k at the same time is proportional to $(P_k^\infty)^2$. From Eq. (8), one gets

$$D(k) = (P_k^\infty)^2/NP(k) \sim k^\nu \tag{9}$$

with

$$\nu = 2(\alpha+1) - \gamma. \tag{10}$$

Then the probability D of finding two walkers on the same node with any degree is given as

$$D = \int_{k_0}^{k_{\max}} D(k)dk \sim \int_{k_0}^{k_{\max}} k^{2(\alpha+1)-\gamma}dk. \tag{11}$$

Since the upper bound $k_{\max}$ diverges with N, the integral $\int_{k_0}^{k_{\max}} k^{2(\alpha+1)-\gamma}dk$ diverges for $\alpha \geq (\gamma-3)/2$. Hence there exists a crossover value α_c given as

$$\alpha_c = (\gamma-3)/2. \tag{12}$$

For $\alpha < \alpha_c$, the lamb survives indefinitely with a finite probability. However, for $\alpha \geq \alpha_c$, the lion captures the lamb with unit probability. To check the scaling relation (10), we measure $D(k)$ on WSFNs with $N=10^5$ and $\langle k\rangle=4$. For 10^5 trials, we count the number $n(k)$ of capture events on nodes with degree k. We obtain $D(k)$ by dividing $n(k)$ by the total trials (10^5). Figure 5 shows the plot of $D(k)$ for several values of α, which scales well with k as Eq. (9). As shown in the insets of Fig. 5, numerical estimates for ν satisfy the relation (10) very well.

To verify the existence of α_c by another method, we now consider the survival probability $S(t)$ of a lamb. $S(t)$ always satisfies $S(t)=S_\infty e^{-t/\tau}$ on random and scale-free networks due to their small world nature [13,26]. As $S(t)=S_\infty e^{-t/\tau}$ in SFNs with any γ, we are interested in the average lifetime $\langle T\rangle$ of a lamb rather than $S(t)$ itself. From $\langle T\rangle=\int_0^\infty t[-dS(t)/dt]dt$ and $S(t)=S_\infty e^{-t/\tau}$, we have $\langle T\rangle \sim \tau$. Hence $\langle T\rangle$ is infinite for $\alpha < \alpha_c$ and finite for $\alpha \geq \alpha_c$ in the limit $N\to\infty$. However, for finite-sized networks, a lamb is eventually captured within N time steps for any α. For $\alpha<\alpha_c$, the maximum lifetime should be on the order of N to guarantee finite survival probability in the limit $N\to\infty$. Hence $\langle T\rangle$ is expected to scale as $\langle T\rangle \sim N$ for $\alpha<\alpha_c$. We measure $\langle T\rangle$ on WSFNs of $\gamma=2.7$ and 3.3 with network size N up to 10^6. From Eq. (12), one reads $\alpha_c=-0.15$ for $\gamma=2.7$ and $\alpha_c=0.15$ for $\gamma=3.3$.

In Fig. 6, we plot $\langle T\rangle$ against N. As shown in each inset, $S(t)$ exponentially decays for any α. For $\gamma=3.3$ [Fig. 6(a)], $\langle T\rangle$ increases with N as N^ϕ with $\phi=0.94(1)$ for $\alpha=0.05$ ($<\alpha_c$) and $\phi=0.90(1)$ for $\alpha=\alpha_c$ (=0.15). We estimate ϕ by measuring successive slopes from the log-log data in Fig. 6(a). For $\alpha=1.0$ ($>\alpha_c$), $\langle T\rangle$ tends to saturate to the asymptotic value $\langle T\rangle_\infty$ with decreasing successive slopes. The exponent ϕ of $\alpha=0.05$ is close to the expected value $\phi=1$. For $\alpha=\alpha_c$ (=0.15), $\langle T\rangle$ seems to diverge with $\phi=0.9$. However, since $\langle T\rangle$ for $\alpha=1.0$ already tends to saturate, it is expected that $\langle T\rangle$ for $\alpha\geq\alpha_c$ would saturate to a finite value in the network with $N>N_c(\alpha)$, where $N_c(\alpha)$ is the character-

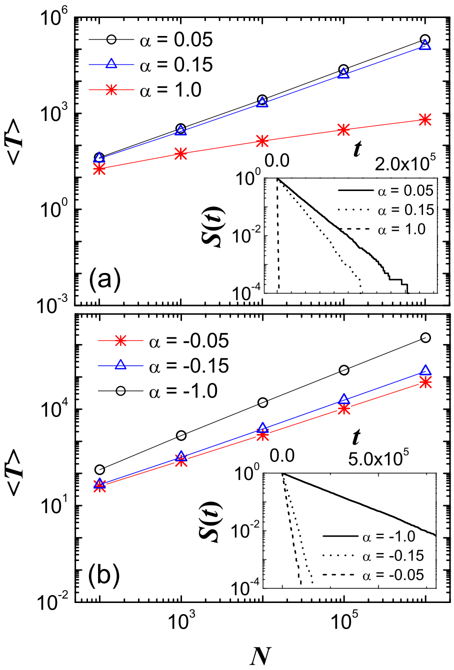

FIG. 6. (Color online) Average lifetime $\langle T \rangle$ of a lamb for $\gamma = 3.3$ (a) and 2.7 (b). The solid line is a guide to the eye. Insets show the semilogarithmic plots of $S(t)$ for $N = 10^5$. From top to bottom, each line corresponds to the $S(t)$ of $\alpha < \alpha_c$, α_c, and $\alpha > \alpha_c$, respectively.

istic size for a given α. For example, for $\alpha = 1.0$, $\langle T \rangle$ does not get into the saturation region even after $N = 10^6$, which implies that $N_c(1.0) > 10^6$ for $\alpha = 1.0$. Since N_c should increase as $\alpha \to \alpha_c$, it is empirically impossible to see the saturation of $\langle T \rangle$ via simulations. Therefore, the initial slope ϕ at α_c may have no special meaning like that of $\alpha > \alpha_c$. The same behavior for $\langle T \rangle$ was observed for the $\alpha = 0$ case [13], where $\langle T \rangle$ initially algebraically increases with continuously varying ϕ (< 1) as $\gamma \to 3$ from below.

For $\gamma = 2.7$ [Fig. 6(b)], we estimate $\phi = 1.00(2)$ for $\alpha = -1.0$ ($< \alpha_c$), as expected. However, for $\alpha = -0.05$ ($> \alpha_c$), $\langle T \rangle$ algebraically increases with $\phi = 0.82(1)$. Since for $\alpha = 0$ [13] N_c is already larger than 10^6 for $\gamma = 2.75$, it is difficult to see the saturation of $\langle T \rangle$. For $\alpha = \alpha_c$, we estimate $\phi = 0.90(1)$. As in $\gamma = 3.3$, the initial slope for $\alpha \geq \alpha_c$ has no special meaning. Based on our numerical results, we are convinced that $\langle T \rangle$ approaches a finite value $\langle T \rangle_\infty$ for $\alpha \geq \alpha_c$ and becomes infinite for $\alpha < \alpha_c$ in the limit $N \to \infty$. Hence in the limit $N \to \infty$, we have

$$\lim_{N \to \infty} S(N,t) = \begin{cases} S_0 e^{-t/\tau_\infty} & (\alpha \geq \alpha_c), \\ S_\infty & (\alpha < \alpha_c), \end{cases} \quad (13)$$

with the characteristic time $\tau_\infty \sim \langle T \rangle_\infty$.

V. AVERAGE MASS OF A NODE WITH DEGREE k

Another interesting quantity in condensation phenomena on networks is the average mass $m(k)$ at a node with degree k in the steady state [13,15,16]. In the ZRP with chipping rate $u(m) \sim m^\delta$, complete condensation takes place for $\delta < \delta_c$, where $\delta_c = 1/(\gamma - 1)$ for unweighted SFNs [15] and $(\alpha+1)/(\gamma-1)$ for WSFNs with the weight (5) [16]. For $\delta < \delta_c$, $m(k)$ increases as $k^{\alpha+1}$ for $k < k_c$, and as $k^{(\alpha+1)/\delta}$ for $k \geq k_c$ on WSFNs. In particular, for $\delta = 0$, $m(k)$ increases as $k^{\alpha+1}$ until $k < k_{\max}$ and jumps to the value $m_{\text{hub}} \approx \rho N$ at $k_{\max}$. Hence condensation takes place at the node with degree $k_{\max}$ in the ZRP.

A recent study of the CA model on unweighted SFNs showed that $m(k)$ linearly increases up to $k_{\max}$ without a jump at $k_{\max}$, unlike in the ZRP with constant chipping rate [13]. The linearity of $m(k)$ comes from the fact that all masses can diffuse. The mass m_{hub} formed on the node with degree $k_{\max}$ can diffuse throughout the network to make the steady state distribution P_i^∞. By taking the average over all nodes, m_{hub} is included in the average mass $m(k)$, unlike in the ZRP where all samples have m_{hub} at $k_{\max}$. The linearity of $m(k)$ on unweighted SFNs results from $P_k^\infty \sim kP(k)$ [13]. Therefore, from $P_k^\infty \sim k^{\alpha+1} P(k)$ on WSFNs, we expect $m(k) \sim k^{\alpha+1}$ up to k_{hub}. To see this explicitly, we derive the relation $m_k \sim k^{\alpha+1}$ as follows.

We consider the average total mass $M(k)$ of nodes with degree k defined as

$$M(k) = \sum_{m=0}^{\infty} m P_\infty(m,k), \quad (14)$$

where $P_\infty(m,k)$ is the probability of finding a walker with mass m at nodes with degree k in the steady state. Since the mass distribution $P(m)$ in the steady state is independent of k, we write $P_\infty(m,k) = P(m) P_k^\infty$. From (7) and (14), one reads

$$M(k) \simeq k^{\alpha+1} P(k) \sum_{m=0}^{\infty} m P(m), \quad (15)$$

where we drop the normalization constant of P_k^∞. Since the number of nodes with degree k is $NP(k)$, $m(k)$ is given as

$$m(k) = \frac{M(k)}{NP(k)} \sim k^{\alpha+1}. \quad (16)$$

To confirm the scaling behavior of $m(k)$, we measure $m(k)$ in the condensed phase on WSFNs of $\gamma = 3.3$ and $N = 10^5$. In Fig. 7, we plot $m(k)$ against k for $\alpha = 0.05$ and 1.0. With $\omega = 1$, we set $\rho = 3.0$ which corresponds to the condensed phase for both α values. Assuming $m(k) \sim k^\eta$, one expects $\eta = 1.05$ for $\alpha = 0.05$ and 2.0 for $\alpha = 1.0$, respectively. From the scaling plot $m(k)/k^\eta$ (inset of Fig. 7), we estimate $\eta = 1.06(2)$ for $\alpha = 0.05$ and $\eta = 1.95(5)$ for $\alpha = 1.0$, which agree well with the predictions.

VI. SUMMARY

In summary, we investigate the properties of the conserved-mass aggregation model on weighted scale-free networks. In WSFNs, the weight w_{ij} is assigned to the link between nodes i and j. We consider the symmetric weight given as $w_{ij} = (k_i k_j)^\alpha$. In the CA model, masses diffuse with unit rate and unit mass chips off from the aggregate mass with rate ω. In addition, the hopping probability T_{ji} from node i to j is given by $T_{ji} = w_{ji} / \sum_{\langle m \rangle} w_{mi}$.

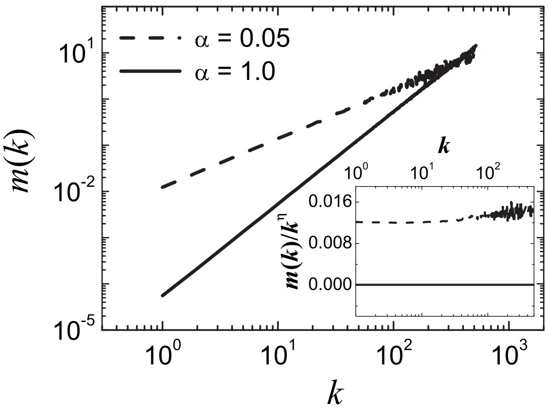

FIG. 7. $m(k)$ for γ=3.3. The solid and the dashed lines correspond to α=1.0 and 0.05, respectively. The inset shows the scaling plot $m(k)k^{-\eta}$ with η=1.06 for α=0.05 (dashed line) and η=1.95 for α=1.0 (solid line).

On WSFNs, a walker finally reaches the hub node with degree $k_{\max}$ for $\alpha \to \infty$, while it is trapped forever at nodes with the minimal degree for $\alpha \to -\infty$. In the lamb-lion capture process, it means that the lion captures the lamb at the hub node within a finite time interval for $\alpha \to \infty$. On the other hand, a lamb survives indefinitely with finite probability for $\alpha \to -\infty$, because the lion cannot escape from a node with the minimal degree to capture a lamb at some other node. Between the two limits, one expects a crossover α_c below which the lifetime $\langle T \rangle$ of a lamb is infinite. However, for $\alpha \geq \alpha_c$, $\langle T \rangle$ is finite. The dependence of $\langle T \rangle$ on α is similar to that on unweighted SFNs of α=0 where $\langle T \rangle$ is infinite for $\gamma > 3$ and finite otherwise [13].

To verify the existence of α_c, we need the stationary probability distribution P_k^∞ of finding a walker at nodes with degree k. From the equation for the transition probability $P_{ji}(t)$ to go from node i to j in t time steps, we analytically find $P_k^\infty \sim k^{\alpha+1-\gamma}$. Next, we consider the so-called lamb-lion capture process. With P_k^∞, we find the probability $D(k)$ of finding two walkers at the same node with degree k at the same time to scale as $D(k) \sim k^{2(\alpha+1)-\gamma}$. Finally, integrating out $D(k)$, we find the death probability D of a lamb. A lamb survives indefinitely with finite survival probability for $\alpha < \alpha_c$, while it is eventually captured by a lion for $\alpha \geq \alpha_c$. We analytically find $\alpha_c = (\gamma - 3)/2$. Therefore, in the limit $N \to \infty$, the lifetime $\langle T \rangle$ of a lamb is finite for $\alpha \geq \alpha_c$, while it is infinite for $\alpha < \alpha_c$. We numerically confirm all analytical results.

The existence of the condensation transitions is known to depend on $\langle T \rangle$ for a lamb [13]. For $\alpha \geq \alpha_c$, $\langle T \rangle$ is finite so condensation always occurs for any nonzero density. On the other hand, for $\alpha < \alpha_c$, infinite $\langle T \rangle$ ensures condensation transitions at a certain critical density ρ_c. For $\alpha \geq \alpha_c$, we numerically confirm that the condensation always takes place at very low density. We also numerically confirm that for $\alpha < \alpha_c$ the CA model on WSFNs undergoes the same type of condensation transitions as those of the SCA model on a regular lattice.

Finally, we investigate the behavior of the average mass $m(k)$ of a node with degree k. In ZRPs with constant chopping rate on networks [15,16], $m(k)$ increases as $k^{\alpha+1}$, and jumps to the total mass of the system at $k_{\max}$. However, in the SCA model on unweighted SFNs, it was shown that $m(k)$ linearly increases with k up to $k_{\max}$ without any jumps [13]. Furthermore, the linearity of $m(k)$ is valid for any $\rho > 0$, which comes from the fact that diffusion is the only relevant physical factor in deciding the distribution $m(k)$. Similarly, on WSFNs, we analytically find and numerically confirm that $m(k)$ algebraically increases as $k^{\alpha+1}$ for any $\rho > 0$ without any jumps.

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